

CONGRUENCES INVOLVING $\binom{2k}{k}^2 \binom{4k}{2k} m^{-k}$

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ABSTRACT. Let $p > 3$ be a prime, and let m be an integer with $p \nmid m$. In the paper, by using the work of Ishii and Deuring's theorem for elliptic curves with complex multiplication we solve some conjectures of Zhi-Wei Sun concerning $\sum_{k=0}^{p-1} \binom{2k}{k}^2 \binom{4k}{2k} m^{-k} \pmod{p^2}$.

MSC: Primary 11A07, Secondary 33C45, 11E25, 11G07, 11L10, 05A10, 05A19

Keywords: Congruence; Legendre polynomial; character sum; elliptic curve; binary quadratic form

1. Introduction.

For positive integers a, b and n , if $n = ax^2 + by^2$ for some integers x and y , we briefly say that $n = ax^2 + by^2$. Let $p > 3$ be a prime. In 2003, Rodriguez-Villegas[RV] posed some conjectures on supercongruences modulo p^2 . One of his conjectures is equivalent to

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{256^k} \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p = x^2 + 2y^2 \equiv 1, 3 \pmod{8}, \\ 0 \pmod{p^2} & \text{if } p \equiv 5, 7 \pmod{8}. \end{cases}$$

This conjecture has been solved by Mortenson[M] and Zhi-Wei Sun[Su2].

Let \mathbb{Z} be the set of integers, and for a prime p let \mathbb{Z}_p be the set of rational numbers whose denominator is coprime to p . Recently the author's brother Zhi-Wei Sun[Su1] posed many conjectures for $\sum_{k=0}^{p-1} \binom{2k}{k}^2 \binom{4k}{2k} m^{-k} \pmod{p^2}$, where $p > 3$ is a prime and $m \in \mathbb{Z}$ with $p \nmid m$. For example, he conjectured (see [Su1, Conjecture A3])

$$(1.1) \quad \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{81^k} \equiv \begin{cases} 0 \pmod{p^2} & \text{if } p \equiv 3, 5, 6 \pmod{7}, \\ 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1, 2, 4 \pmod{7} \text{ and so } p = x^2 + 7y^2. \end{cases}$$

Let $\{P_n(x)\}$ be the Legendre polynomials given by (see [MOS, pp. 228-232], [G, (3.132)-(3.133)])

$$(1.2) \quad P_n(x) = \frac{1}{2^n} \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{k} (-1)^k \binom{2n-2k}{n} x^{n-2k} = \frac{1}{2^n \cdot n!} \cdot \frac{d^n}{dx^n} (x^2 - 1)^n,$$

where $[a]$ is the greatest integer not exceeding a . From (1.2) we see that

$$(1.3) \quad P_n(-x) = (-1)^n P_n(x).$$

Let $(\frac{a}{m})$ be the Jacobi symbol. For a prime $p > 3$, In [S2] the author showed that

$$(1.4) \quad P_{[\frac{p}{4}]}(t) \equiv \sum_{k=0}^{[p/4]} \binom{4k}{2k} \binom{2k}{k} \left(\frac{1-t}{128}\right)^k \equiv -\left(\frac{6}{p}\right) \sum_{x=0}^{p-1} \left(\frac{x^3 - \frac{3(3t+5)}{2}x + 9t + 7}{p}\right) \pmod{p}.$$

We note that $p \mid \binom{2k}{k} \binom{4k}{2k}$ for $\frac{p}{4} < k < p$.

Let $p > 3$ be a prime, $m \in \mathbb{Z}_p$, $m \not\equiv 0 \pmod{p}$ and $t = \sqrt{1 - 256/m}$. In [S2] the author showed that

$$(1.5) \quad \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{m^k} \equiv P_{[\frac{p}{4}]}(t)^2 \pmod{p}.$$

In the paper we show that

$$(1.6) \quad \sum_{k=0}^{p-1} \binom{2k}{k}^2 \binom{4k}{2k} (x(1 - 64x))^k \equiv \left(\sum_{k=0}^{p-1} \binom{2k}{k} \binom{4k}{2k} x^k \right)^2 \pmod{p^2}.$$

On the basis of (1.5) and (1.6), we prove some congruences involving $\sum_{k=0}^{p-1} \binom{2k}{k}^2 \binom{4k}{2k} m^{-k}$ in the cases

$$m = 81, 648, 12^4, 28^4, 1584^2, 396^4, -144, -3969, -2^{10} \cdot 3^4, -2^{10} \cdot 21^4, -3 \cdot 2^{12}, -2^{14} \cdot 3^4 \cdot 5.$$

Thus we partially solve some conjectures posed by Zhi-Wei Sun in [Su1]. As examples, we partially solve (1.1), and show that for primes $p \equiv \pm 1 \pmod{8}$,

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{1584^{2k}} \equiv \begin{cases} 4x^2 \pmod{p} & \text{if } (\frac{p}{11}) = 1 \text{ and so } p = x^2 + 22y^2, \\ 0 \pmod{p^2} & \text{if } (\frac{p}{11}) = -1. \end{cases}$$

2. A general congruence modulo p^2 .

Lemma 2.1. *Let m be a nonnegative integer. Then*

$$\sum_{k=0}^m \binom{2k}{k}^2 \binom{4k}{2k} \binom{k}{m-k} (-64)^{m-k} = \sum_{k=0}^m \binom{2k}{k} \binom{4k}{2k} \binom{2(m-k)}{m-k} \binom{4(m-k)}{2(m-k)}.$$

We prove the lemma by using WZ method and Mathematica. Clearly the result is true for $m = 0, 1$. Since both sides satisfy the same recurrence relation

$$1024(m+1)(2m+1)(2m+3)S(m) - 8(2m+3)(8m^2 + 24m + 19)S(m+1) + (m+2)^3 S(m+2) = 0,$$

we see that Lemma 2.1 is true. The proof certificate for the left hand side is

$$-\frac{4096k^2(m+2)(m-2k)(m-2k+1)}{(m-k+1)(m-k+2)},$$

and the proof certificate for the right hand side is

$$\frac{16k^2(4m-4k+1)(4m-4k+3)(16m^2-16mk+55m-26k+46)}{(m-k+1)^2(m-k+2)^2}.$$

Theorem 2.1. *Let p be an odd prime and let x be a variable. Then*

$$\sum_{k=0}^{p-1} \binom{2k}{k}^2 \binom{4k}{2k} (x(1-64x))^k \equiv \left(\sum_{k=0}^{p-1} \binom{2k}{k} \binom{4k}{2k} x^k \right)^2 \pmod{p^2}.$$

Proof. It is clear that

$$\begin{aligned} & \sum_{k=0}^{p-1} \binom{2k}{k}^2 \binom{4k}{2k} (x(1-64x))^k \\ &= \sum_{k=0}^{p-1} \binom{2k}{k}^2 \binom{4k}{2k} x^k \sum_{r=0}^k \binom{k}{r} (-64x)^r \\ &= \sum_{m=0}^{2(p-1)} x^m \sum_{k=0}^{\min\{m, p-1\}} \binom{2k}{k}^2 \binom{4k}{2k} \binom{k}{m-k} (-64)^{m-k}. \end{aligned}$$

Suppose $p \leq m \leq 2p-2$ and $0 \leq k \leq p-1$. If $k > \frac{p}{2}$, then $p \mid \binom{2k}{k}$ and so $p^2 \mid \binom{2k}{k}^2$. If $k < \frac{p}{2}$, then $m-k \geq p-k > k$ and so $\binom{k}{m-k} = 0$. Thus, from the above and Lemma 2.1 we deduce

$$\begin{aligned} & \sum_{k=0}^{p-1} \binom{2k}{k}^2 \binom{4k}{2k} (x(1-64x))^k \\ &\equiv \sum_{m=0}^{p-1} x^m \sum_{k=0}^m \binom{2k}{k}^2 \binom{4k}{2k} \binom{k}{m-k} (-64)^{m-k} \\ &= \sum_{m=0}^{p-1} x^m \sum_{k=0}^m \binom{2k}{k} \binom{4k}{2k} \binom{2(m-k)}{m-k} \binom{4(m-k)}{2(m-k)} \\ &= \sum_{k=0}^{p-1} \binom{2k}{k} \binom{4k}{2k} x^k \sum_{m=k}^{p-1} \binom{2(m-k)}{m-k} \binom{4(m-k)}{2(m-k)} x^{m-k} \\ &= \sum_{k=0}^{p-1} \binom{2k}{k} \binom{4k}{2k} x^k \sum_{r=0}^{p-1-k} \binom{2r}{r} \binom{4r}{2r} x^r \\ &= \sum_{k=0}^{p-1} \binom{2k}{k} \binom{4k}{2k} x^k \left(\sum_{r=0}^{p-1} \binom{2r}{r} \binom{4r}{2r} x^r - \sum_{r=p-k}^{p-1} \binom{2r}{r} \binom{4r}{2r} x^r \right) \\ &= \left(\sum_{k=0}^{p-1} \binom{2k}{k} \binom{4k}{2k} x^k \right)^2 - \sum_{k=0}^{p-1} \binom{2k}{k} \binom{4k}{2k} x^k \sum_{r=p-k}^{p-1} \binom{2r}{r} \binom{4r}{2r} x^r \pmod{p^2}. \end{aligned}$$

Now suppose $0 \leq k \leq p-1$ and $p-k \leq r \leq p-1$. If $k \geq \frac{3p}{4}$, then $p^2 \nmid (2k)!$, $p^3 \mid (4k)!$ and so $\binom{2k}{k} \binom{4k}{2k} = \frac{(4k)!}{(2k)!k!^2} \equiv 0 \pmod{p^2}$. If $k < \frac{p}{4}$, then $r \geq p-k \geq \frac{3p}{4}$ and so $\binom{2r}{r} \binom{4r}{2r} = \frac{(4r)!}{(2r)!r!^2} \equiv 0 \pmod{p^2}$. If $\frac{p}{4} < k < \frac{p}{2}$, then $r \geq p-k > \frac{p}{2}$, $p \nmid (2k)!$, $p \mid (4k)!$,

$p \mid \binom{2r}{r}$ and $\binom{2k}{k} \binom{4k}{2k} = \frac{(4k)!}{(2k)!k!2} \equiv 0 \pmod{p}$. If $\frac{p}{2} < k < \frac{3p}{4}$, then $r \geq p - k > \frac{p}{4}$, $p \mid \binom{2k}{k}$ and $\binom{2r}{r} \binom{4r}{2r} = \frac{(4r)!}{(2r)!r!2} \equiv 0 \pmod{p}$. Hence we always have $\binom{2k}{k} \binom{4k}{2k} \binom{2r}{r} \binom{4r}{2r} \equiv 0 \pmod{p^2}$ and so

$$\sum_{k=0}^{p-1} \binom{2k}{k} \binom{4k}{2k} x^k \sum_{r=p-k}^{p-1} \binom{2r}{r} \binom{4r}{2r} x^r \equiv 0 \pmod{p^2}.$$

Now combining all the above we obtain the result.

Corollary 2.1. *Let $p > 3$ be a prime and $m \in \mathbb{Z}_p$ with $m \not\equiv 0 \pmod{p}$. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{m^k} \equiv \left(\sum_{k=0}^{p-1} \binom{2k}{k} \binom{4k}{2k} \left(\frac{1 - \sqrt{1 - 256/m}}{128} \right)^k \right)^2 \pmod{p^2}.$$

Proof. Taking $x = \frac{1 - \sqrt{1 - 256/m}}{128}$ in Theorem 2.1 we deduce the result.

Corollary 2.2. *Let $p > 3$ be a prime and $m \in \mathbb{Z}_p$ with $m \not\equiv 0, 256 \pmod{p}$. Then*

$$\sum_{k=0}^{[p/4]} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{m^k} \equiv 0 \pmod{p} \quad \text{implies} \quad \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{m^k} \equiv 0 \pmod{p^2}.$$

Proof. For $\frac{p}{4} < k < p$ we see that $\binom{2k}{k}^2 \binom{4k}{2k} = \frac{(4k)!}{k!^4} \equiv 0 \pmod{p}$. Suppose $\sum_{k=0}^{[p/4]} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{m^k} \equiv 0 \pmod{p}$. Then

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{m^k} \equiv \sum_{k=0}^{[p/4]} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{m^k} \equiv 0 \pmod{p}.$$

Using Corollary 2.1 we see that

$$\sum_{k=0}^{p-1} \binom{2k}{k} \binom{4k}{2k} \left(\frac{1 - \sqrt{1 - 256/m}}{128} \right)^k \equiv 0 \pmod{p}.$$

Thus the result follows from Corollary 2.1.

Corollary 2.3. *Let $p \equiv 1, 3 \pmod{8}$ be a prime and $p = c^2 + 2d^2$ with $c, d \in \mathbb{Z}$ and $c \equiv 1 \pmod{4}$. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{4k}{2k}}{128^k} \equiv (-1)^{[\frac{p}{8}] + \frac{p-1}{2}} \left(2c - \frac{p}{2c} \right) \pmod{p^2}.$$

Proof. By [S2, Theorem 2.1] we have

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{4k}{2k}}{128^k} \equiv \sum_{k=0}^{[p/4]} \frac{\binom{2k}{k} \binom{4k}{2k}}{128^k} \equiv (-1)^{[\frac{p}{8}] + \frac{p-1}{2}} 2c \pmod{p}.$$

Set $\sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{4k}{2k}}{128^k} = (-1)^{[\frac{p}{8}] + \frac{p-1}{2}} 2c + qp$. Then

$$\left(\sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{4k}{2k}}{128^k} \right)^2 = ((-1)^{[\frac{p}{8}] + \frac{p-1}{2}} 2c + qp)^2 \equiv 4c^2 + (-1)^{[\frac{p}{8}] + \frac{p-1}{2}} 4cqp \pmod{p^2}.$$

Taking $x = \frac{1}{128}$ in Theorem 2.1 we get

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{256^k} \equiv \left(\sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{4k}{2k}}{128^k} \right)^2 \pmod{p^2}.$$

From [M] and [Su2] we have

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{256^k} \equiv 4c^2 - 2p \pmod{p^2}.$$

Thus

$$4c^2 - 2p \equiv \left(\sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{4k}{2k}}{128^k} \right)^2 \equiv 4c^2 + (-1)^{[\frac{p}{8}] + \frac{p-1}{2}} 4cqp \pmod{p^2}$$

and hence $q \equiv -(-1)^{[\frac{p}{8}] + \frac{p-1}{2}} \frac{1}{2c} \pmod{p}$. So the corollary is proved.

3. Congruences for $\sum_{k=0}^{p-1} \binom{2k}{k}^2 \binom{4k}{2k} m^{-k}$.

Let $p > 3$ be a prime and $m \in \mathbb{Z}$ with $p \nmid m$. In the section we partially solve Z.W. Sun's conjectures on $\sum_{k=0}^{p-1} \binom{2k}{k}^2 \binom{4k}{2k} m^{-k} \pmod{p^2}$.

Lemma 3.1 ([S2, (4.2)]). *Let $p > 3$ be a prime and let u be a variable. Then*

$$P_{[\frac{p}{4}]}(u) \equiv -\left(\frac{6}{p}\right) \sum_{x=0}^{p-1} \left(x^3 - \frac{3}{2}(3u+5)x + 9u+7 \right)^{\frac{p-1}{2}} \pmod{p}.$$

Lemma 3.2 ([S2, Theorem 4.1]). *Let p be an odd prime, $m \in \mathbb{Z}_p$, $m \not\equiv 0 \pmod{p}$ and $t = \sqrt{1 - 256/m}$. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{m^k} \equiv P_{[\frac{p}{4}]}(t)^2 \equiv \left(\sum_{x=0}^{p-1} (x^3 + 4x^2 + (2-2t)x)^{\frac{p-1}{2}} \right)^2 \pmod{p}.$$

Lemma 3.3. *Let p be an odd prime, $m \in \mathbb{Z}_p$, $m \not\equiv 0 \pmod{p}$ and $t = \sqrt{1 - 256/m}$. If $P_{[\frac{p}{4}]}(t) \equiv 0 \pmod{p}$, then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{m^k} \equiv 0 \pmod{p^2}.$$

Proof. By Corollary 2.1 we have

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{m^k} \equiv \left(\sum_{k=0}^{p-1} \binom{2k}{k} \binom{4k}{2k} \left(\frac{1-t}{128} \right)^k \right)^2 \pmod{p^2}.$$

Observe that $p \mid \binom{2k}{k} \binom{4k}{2k}$ for $\frac{p}{4} < k < p$. From [S2, Lemma 2.2] we see that

$$\sum_{k=0}^{p-1} \binom{2k}{k} \binom{4k}{2k} \left(\frac{1-t}{128} \right)^k \equiv P_{[\frac{p}{4}]}(t) \pmod{p}.$$

Thus the result follows.

Lemma 3.4 ([S3, Lemma 4.1]). *Let p be an odd prime and let a, m, n be p -adic integers. Then*

$$\sum_{x=0}^{p-1} (x^3 + a^2mx + a^3n)^{\frac{p-1}{2}} \equiv a^{\frac{p-1}{2}} \sum_{x=0}^{p-1} (x^3 + mx + n)^{\frac{p-1}{2}} \pmod{p}.$$

Moreover, if a, m, n are congruent to some integers, then

$$\sum_{x=0}^{p-1} \left(\frac{x^3 + a^2mx + a^3n}{p} \right) = \left(\frac{a}{p} \right) \sum_{x=0}^{p-1} \left(\frac{x^3 + mx + n}{p} \right).$$

Theorem 3.1. *Let $p \neq 2, 3, 7$ be a prime. Then*

$$\begin{aligned} & P_{[\frac{p}{4}]} \left(\frac{5\sqrt{-7}}{9} \right) \\ & \equiv \begin{cases} -\left(\frac{3(7+\sqrt{-7})}{p} \right) \left(\frac{C}{7} \right) 2C \pmod{p} & \text{if } p \equiv 1, 2, 4 \pmod{7} \text{ and so } p = C^2 + 7D^2, \\ 0 \pmod{p} & \text{if } p \equiv 3, 5, 6 \pmod{7} \end{cases} \end{aligned}$$

and

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{81^k} \equiv \begin{cases} 4C^2 \pmod{p} & \text{if } p \equiv 1, 2, 4 \pmod{7} \text{ and so } p = C^2 + 7D^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 3, 5, 6 \pmod{7}. \end{cases}$$

Proof. By Lemma 3.1 we have

$$P_{[\frac{p}{4}]} \left(\frac{5\sqrt{-7}}{9} \right) \equiv -\left(\frac{6}{p} \right) \sum_{x=0}^{p-1} \left(x^3 - \frac{5}{2}(3 + \sqrt{-7})x + 7 + 5\sqrt{-7} \right)^{\frac{p-1}{2}} \pmod{p}.$$

Since

$$\frac{-\frac{5}{2}(3 + \sqrt{-7})}{-35} = \left(\frac{1 - \sqrt{-7}}{2\sqrt{-7}} \right)^2 \quad \text{and} \quad \frac{7 + 5\sqrt{-7}}{-98} = \left(\frac{1 - \sqrt{-7}}{2\sqrt{-7}} \right)^3,$$

by the above and Lemma 3.4 we have

$$P_{[\frac{p}{4}]} \left(\frac{5\sqrt{-7}}{9} \right) \equiv - \left(\frac{6}{p} \right) \left(\frac{1 - \sqrt{-7}}{2\sqrt{-7}} \right)^{\frac{p-1}{2}} \sum_{x=0}^{p-1} \left(\frac{x^3 - 35x - 98}{p} \right) \pmod{p}.$$

Observe that $(x+7)^3 - 35(x+7) - 98 = x^3 + 21x^2 + 112x$. By the work of Rajwade ([R1,R2]), we get

$$(3.1) \quad \sum_{x=0}^{p-1} \left(\frac{x^3 - 35x - 98}{p} \right) = \sum_{x=0}^{p-1} \left(\frac{x^3 + 21x^2 + 112x}{p} \right) \\ = \begin{cases} 2C(\frac{C}{7}) & \text{if } p = C^2 + 7D^2 \equiv 1, 2, 4 \pmod{7}, \\ 0 & \text{if } p \equiv 3, 5, 6 \pmod{7}. \end{cases}$$

For $p \equiv 1, 2, 4 \pmod{7}$ we see that

$$\left(\frac{6}{p} \right) \left(\frac{1 - \sqrt{-7}}{2\sqrt{-7}} \right)^{\frac{p-1}{2}} = \left(\frac{6}{p} \right) \left(\frac{7 + \sqrt{-7}}{2 \cdot (-7)} \right)^{\frac{p-1}{2}} \equiv \left(\frac{3}{p} \right) \left(\frac{7 + \sqrt{-7}}{p} \right) \pmod{p}.$$

Thus, from the above we deduce the congruence for $P_{[\frac{p}{4}]}(\frac{5\sqrt{-7}}{9}) \pmod{p}$. Applying Lemmas 3.2 and 3.3 we obtain the remaining result.

Let $p > 3$ be a prime and let \mathbb{F}_p be the field of p elements. For $m, n \in \mathbb{F}_p$ let $\#E_p(x^3 + mx + n)$ be the number of points on the curve $E_p: y^2 = x^3 + mx + n$ over the field \mathbb{F}_p . It is well known that

$$(3.2) \quad \#E_p(x^3 + mx + n) = p + 1 + \sum_{x=0}^{p-1} \left(\frac{x^3 + mx + n}{p} \right).$$

Let $K = \mathbb{Q}(\sqrt{-d})$ be an imaginary quadratic field and the curve $y^2 = x^3 + mx + n$ has complex multiplication by K . By Deuring's theorem ([C, Theorem 14.16],[PV],[I]), we have

$$(3.3) \quad \#E_p(x^3 + mx + n) = \begin{cases} p + 1 & \text{if } p \text{ is inert in } K, \\ p + 1 - \pi - \bar{\pi} & \text{if } p = \pi\bar{\pi} \text{ in } K, \end{cases}$$

where π is in an order in K and $\bar{\pi}$ is the conjugate number of π . If $4p = u^2 + dv^2$ with $u, v \in \mathbb{Z}$, we may take $\pi = \frac{1}{2}(u + v\sqrt{-d})$. Thus,

$$(3.4) \quad \sum_{x=0}^{p-1} \left(\frac{x^3 + mx + n}{p} \right) = \begin{cases} \pm u & \text{if } 4p = u^2 + dv^2 \text{ with } u, v \in \mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases}$$

In [JM] and [PV] the sign of u in (3.4) was determined for those imaginary quadratic fields K with class number 1. In [LM] and [I] the sign of u in (3.4) was determined for imaginary quadratic fields K with class number 2.

Theorem 3.2. *Let p be a prime such that $p \equiv \pm 1 \pmod{12}$. Then*

$$P_{[\frac{p}{4}]} \left(\frac{7}{12} \sqrt{3} \right) \equiv \begin{cases} \left(\frac{2+2\sqrt{3}}{p} \right) 2x \pmod{p} & \text{if } p = x^2 + 9y^2 \equiv 1 \pmod{12} \text{ with } 3 \mid x-1, \\ 0 \pmod{p} & \text{if } p \equiv 11 \pmod{12} \end{cases}$$

and

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{(-12288)^k} \equiv \begin{cases} 4x^2 \pmod{p} & \text{if } p = x^2 + 9y^2 \equiv 1 \pmod{12}, \\ 0 \pmod{p^2} & \text{if } p \equiv 11 \pmod{12}. \end{cases}$$

Proof. From [I, p.133] we know that the elliptic curve defined by the equation $y^2 = x^3 - (120 + 42\sqrt{3})x + 448 + 336\sqrt{3}$ has complex multiplication by the order of discriminant -36 . Thus, by (3.4) and [I, Theorem 3.1] we have

$$\begin{aligned} & \sum_{n=0}^{p-1} \left(\frac{n^3 - (120 + 42\sqrt{3})x + 448 + 336\sqrt{3}}{p} \right) \\ &= \begin{cases} -2x \left(\frac{1+\sqrt{3}}{p} \right) & \text{if } p = x^2 + 9y^2 \equiv 1 \pmod{12} \text{ with } 3 \mid x-1, \\ 0 & \text{if } p \equiv 11 \pmod{12}. \end{cases} \end{aligned}$$

By Lemma 3.1 we have

$$\begin{aligned} P_{[\frac{p}{4}]} \left(\frac{7}{12} \sqrt{3} \right) &\equiv - \left(\frac{6}{p} \right) \sum_{n=0}^{p-1} \left(n^3 - \frac{60 + 21\sqrt{3}}{8} n + \frac{28 + 21\sqrt{3}}{4} \right)^{\frac{p-1}{2}} \\ &\equiv - \left(\frac{6}{p} \right) \sum_{n=0}^{p-1} \left(\left(\frac{n}{4} \right)^3 - \frac{60 + 21\sqrt{3}}{8} \cdot \frac{n}{4} + \frac{28 + 21\sqrt{3}}{4} \right)^{\frac{p-1}{2}} \\ &\equiv - \left(\frac{6}{p} \right) \sum_{n=0}^{p-1} \left(\frac{n^3 - (120 + 42\sqrt{3})x + 448 + 336\sqrt{3}}{p} \right) \pmod{p}. \end{aligned}$$

Now combining all the above we obtain the congruence for $P_{[\frac{p}{4}]} \left(\frac{7}{12} \sqrt{3} \right) \pmod{p}$. Applying Lemmas 3.2 and 3.3 we deduce the remaining result.

Remark 3.1 In [Su1, Conjecture A24], Z.W. Sun conjectured that for any prime $p > 3$,

$$\begin{aligned} & \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{(-12288)^k} \\ &\equiv \begin{cases} (-1)^{[\frac{p}{6}]} (4x^2 - 2p) \pmod{p} & \text{if } p = x^2 + y^2 \equiv 1 \pmod{12} \text{ and } 4 \mid x-1, \\ -4 \left(\frac{xy}{3} \right) xy \pmod{p^2} & \text{if } p = x^2 + y^2 \equiv 5 \pmod{12} \text{ and } 4 \mid x-1, \\ 0 \pmod{p^2} & \text{if } p \equiv 3 \pmod{4}. \end{cases} \end{aligned}$$

If p is a prime such that $p = x^2 + 5y^2 \equiv 1, 9 \pmod{20}$, by using [LM, Theorem 11] the author proved in [S2, Theorem 4.7] that

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{(-1024)^k} \equiv 4x^2 \pmod{p}.$$

Now we give similar results concerning $x^2 + 13y^2$ and $x^2 + 37y^2$.

Theorem 3.3. *Let p be an odd prime such that $p \neq 3$ and $(\frac{13}{p}) = 1$. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{(-82944)^k} \equiv \begin{cases} 4x^2 \pmod{p} & \text{if } p = x^2 + 13y^2 \equiv 1 \pmod{4}, \\ 0 \pmod{p^2} & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

Proof. From [LM, Table II] we know that the elliptic curve defined by the equation $y^2 = x^3 + 4x^2 + (2 - \frac{5}{9}\sqrt{13})x$ has complex multiplication by the order of discriminant -52 . Thus, by (3.4) we have

$$\begin{aligned} & \sum_{n=0}^{p-1} \left(\frac{n^3 + 4n^2 + (2 - \frac{5}{9}\sqrt{13})n}{p} \right) \\ &= \begin{cases} 2x & \text{if } p \equiv 1 \pmod{4} \text{ and so } p = x^2 + 13y^2, \\ 0 & \text{if } p \equiv 3 \pmod{4}. \end{cases} \end{aligned}$$

Now taking $m = -2^{10} \cdot 3^4$ and $t = \frac{5}{18}\sqrt{13}$ in Lemmas 3.2 and 3.3 and applying the above we deduce the result.

Remark 3.2 Let $p \neq 3, 13$ be an odd prime. In [Su1, Conjecture A17], Z.W. Sun conjectured that

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{(-82944)^k} \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } (\frac{13}{p}) = (\frac{-1}{p}) = 1 \text{ and so } p = x^2 + 13y^2, \\ 2p - 2x^2 \pmod{p^2} & \text{if } (\frac{13}{p}) = (\frac{-1}{p}) = -1 \text{ and so } 2p = x^2 + 13y^2, \\ 0 \pmod{p^2} & \text{if } (\frac{13}{p}) = -(\frac{-1}{p}). \end{cases}$$

Theorem 3.4. *Let p be an odd prime such that $p \neq 3, 7$ and $(\frac{37}{p}) = 1$. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{(-2^{10} \cdot 21^4)^k} \equiv \begin{cases} 4x^2 \pmod{p} & \text{if } p \equiv 1 \pmod{4} \text{ and so } p = x^2 + 37y^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

Proof. From [LM, Table II] we know that the elliptic curve defined by the equation $y^2 = x^3 + 4x^2 + (2 - \frac{145}{441}\sqrt{37})x$ has complex multiplication by the order of discriminant -148 . Thus, by (3.4) we have

$$\begin{aligned} & \sum_{n=0}^{p-1} \left(\frac{n^3 + 4n^2 + (2 - \frac{145}{441}\sqrt{37})n}{p} \right) \\ &= \begin{cases} 2x & \text{if } p \equiv 1 \pmod{4} \text{ and so } p = x^2 + 37y^2, \\ 0 & \text{if } p \equiv 3 \pmod{4}. \end{cases} \end{aligned}$$

Now taking $m = -2^{10} \cdot 21^4$ and $t = \frac{145}{882}\sqrt{37}$ in Lemmas 3.2 and 3.3 and applying the above we deduce the result.

Remark 3.3 Let $p \neq 3, 7, 37$ be a prime. In [Su1, Conjecture A19], Z.W. Sun conjectured that

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{(-2^{10} \cdot 21^4)^k} \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{37}{p}\right) = \left(\frac{-1}{p}\right) = 1 \text{ and so } p = x^2 + 37y^2, \\ 2p - 2x^2 \pmod{p^2} & \text{if } \left(\frac{37}{p}\right) = \left(\frac{-1}{p}\right) = -1 \text{ and so } 2p = x^2 + 37y^2, \\ 0 \pmod{p^2} & \text{if } \left(\frac{37}{p}\right) = -\left(\frac{-1}{p}\right). \end{cases}$$

Let $b \in \{3, 5, 11, 29\}$ and $f(b) = 48^2, 12^4, 1584^2, 396^4$ according as $b = 3, 5, 11, 29$. For any odd prime p with $p \nmid bf(b)$, Z.W. Sun conjectured that ([Su1, Conjectures A14, A16, A18 and A21])

$$(3.5) \quad \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{f(b)^k} \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{2}{p}\right) = \left(\frac{-b}{p}\right) = 1 \text{ and so } p = x^2 + 2by^2, \\ 2p - 8x^2 \pmod{p^2} & \text{if } \left(\frac{2}{p}\right) = \left(\frac{-b}{p}\right) = -1 \text{ and so } p = 2x^2 + by^2, \\ 0 \pmod{p^2} & \text{if } \left(\frac{2}{p}\right) = -\left(\frac{-b}{p}\right). \end{cases}$$

Now we partially solve the above conjecture.

Theorem 3.5. *Let p be an odd prime such that $p \equiv \pm 1 \pmod{8}$. Then*

$$P_{[\frac{p}{4}]} \left(\frac{2\sqrt{2}}{3} \right) \equiv \begin{cases} (-1)^{\frac{p-1}{2}} \left(\frac{\sqrt{2}}{p} \right) \left(\frac{x}{3} \right) 2x \pmod{p} & \text{if } p = x^2 + 6y^2 \equiv 1, 7 \pmod{24}, \\ 0 \pmod{p} & \text{if } p \equiv 17, 23 \pmod{24} \end{cases}$$

and

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{48^{2k}} \equiv \begin{cases} 4x^2 \pmod{p} & \text{if } p = x^2 + 6y^2 \equiv 1, 7 \pmod{24}, \\ 0 \pmod{p^2} & \text{if } p \equiv 17, 23 \pmod{24}. \end{cases}$$

Proof. From [I, p.133] we know that the elliptic curve defined by the equation $y^2 = x^3 + (-21 + 12\sqrt{2})x - 28 + 22\sqrt{2}$ has complex multiplication by the order of discriminant -24 . Thus, by (3.4) and [I, Theorem 3.1] we have

$$\begin{aligned} & \sum_{n=0}^{p-1} \left(\frac{n^3 + (-21 + 12\sqrt{2})n - 28 + 22\sqrt{2}}{p} \right) \\ &= \begin{cases} 2x \left(\frac{2x}{3} \right) \left(\frac{1+\sqrt{2}}{p} \right) & \text{if } p \equiv 1, 7 \pmod{24} \text{ and so } p = x^2 + 6y^2, \\ 0 & \text{if } p \equiv 17, 23 \pmod{24}. \end{cases} \end{aligned}$$

By Lemma 3.1 we have

$$P_{[\frac{p}{4}]} \left(\frac{2\sqrt{2}}{3} \right) \equiv - \left(\frac{6}{p} \right) \sum_{n=0}^{p-1} \left(n^3 - \frac{15 + 6\sqrt{2}}{2}n + 7 + 6\sqrt{2} \right)^{\frac{p-1}{2}} \pmod{p}.$$

Since

$$\frac{-(15 + 6\sqrt{2})/2}{-21 + 12\sqrt{2}} = \left(\frac{\sqrt{2} + 1}{\sqrt{2}} \right)^2 \quad \text{and} \quad \frac{7 + 6\sqrt{2}}{-28 + 22\sqrt{2}} = \left(\frac{\sqrt{2} + 1}{\sqrt{2}} \right)^3,$$

by Lemma 3.4 and the above we have

$$\begin{aligned} P_{[\frac{p}{4}]} \left(\frac{2\sqrt{2}}{3} \right) &\equiv - \left(\frac{6}{p} \right) \left(\frac{\sqrt{2}(\sqrt{2}+1)}{p} \right) \sum_{n=0}^{p-1} \left(\frac{n^3 + (-21 + 12\sqrt{2})n - 28 + 22\sqrt{2}}{p} \right) \\ &\equiv \begin{cases} - \left(\frac{6}{p} \right) \left(\frac{\sqrt{2}}{p} \right) 2x \left(\frac{2x}{3} \right) \pmod{p} & \text{if } p = x^2 + 6y^2 \equiv 1, 7 \pmod{24}, \\ 0 \pmod{p} & \text{if } p \equiv 17, 23 \pmod{24}. \end{cases} \end{aligned}$$

This yields the result for $P_{[\frac{p}{4}]} \left(\frac{2\sqrt{2}}{3} \right) \pmod{p}$. Taking $m = 48^2$ and $t = \frac{2}{3}\sqrt{2}$ in Lemmas 3.2 and 3.3 and applying the above we deduce the remaining result.

Theorem 3.6. *Let p be a prime such that $p \equiv \pm 1 \pmod{5}$. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{12^{4k}} \equiv \begin{cases} 4x^2 \pmod{p} & \text{if } p = x^2 + 10y^2 \equiv 1, 9, 11, 19 \pmod{40}, \\ 0 \pmod{p^2} & \text{if } p \equiv 21, 29, 31, 39 \pmod{40}. \end{cases}$$

Proof. From [LM, Table II] we know that the elliptic curve defined by the equation $y^2 = x^3 + 4x^2 + (2 - \frac{8}{9}\sqrt{5})x$ has complex multiplication by the order of discriminant -40 . Thus, by (3.4) we have

$$\begin{aligned} &\sum_{n=0}^{p-1} \left(\frac{n^3 + 4n^2 + (2 - \frac{8}{9}\sqrt{5})n}{p} \right) \\ &= \begin{cases} 2x & \text{if } p \equiv 1, 9, 11, 19 \pmod{40} \text{ and so } p = x^2 + 10y^2, \\ 0 & \text{if } p \equiv 21, 29, 31, 39 \pmod{40}. \end{cases} \end{aligned}$$

Now taking $m = 12^4$ and $t = \frac{4}{9}\sqrt{5}$ in Lemmas 3.2 and 3.3 and applying the above we deduce the result.

Theorem 3.7. *Let p be a prime such that $p \equiv \pm 1 \pmod{8}$. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{1584^{2k}} \equiv \begin{cases} 4x^2 \pmod{p} & \text{if } \left(\frac{p}{11} \right) = 1 \text{ and so } p = x^2 + 22y^2, \\ 0 \pmod{p^2} & \text{if } \left(\frac{p}{11} \right) = -1. \end{cases}$$

Proof. From [LM, Table II] we know that the elliptic curve defined by the equation $y^2 = x^3 + 4x^2 + (2 - \frac{140}{99}\sqrt{2})x$ has complex multiplication by the order of discriminant -88 . Thus, by (3.4) we have

$$\begin{aligned} &\sum_{n=0}^{p-1} \left(\frac{n^3 + 4n^2 + (2 - \frac{140}{99}\sqrt{2})n}{p} \right) \\ &= \begin{cases} 2x & \text{if } \left(\frac{p}{11} \right) = 1 \text{ and so } p = x^2 + 22y^2, \\ 0 & \text{if } \left(\frac{p}{11} \right) = -1. \end{cases} \end{aligned}$$

Now taking $m = 1584^2$ and $t = \frac{70}{99}\sqrt{2}$ in Lemmas 3.2 and 3.3 and applying the above we deduce the result.

Theorem 3.8. *Let p be an odd prime such that $(\frac{29}{p}) = 1$. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{396^{4k}} \equiv \begin{cases} 4x^2 \pmod{p} & \text{if } p = x^2 + 58y^2 \equiv 1, 3 \pmod{8}, \\ 0 \pmod{p^2} & \text{if } p \equiv 5, 7 \pmod{8}. \end{cases}$$

Proof. From [LM, Table II] we know that the elliptic curve defined by the equation $y^2 = x^3 + 4x^2 + (2 - \frac{3640}{9801}\sqrt{29})x$ has complex multiplication by the order of discriminant -232 . Thus, by (3.4) we have

$$\begin{aligned} & \sum_{n=0}^{p-1} \left(\frac{n^3 + 4n^2 + (2 - \frac{3640}{9801}\sqrt{29})n}{p} \right) \\ &= \begin{cases} 2x & \text{if } p \equiv 1, 3 \pmod{8} \text{ and so } p = x^2 + 58y^2, \\ 0 & \text{if } p \equiv 5, 7 \pmod{8}. \end{cases} \end{aligned}$$

Now taking $m = 396^4$ and $t = \frac{1820}{9801}\sqrt{29}$ in Lemmas 3.2 and 3.3 and applying the above we deduce the result.

Theorem 3.9. *Let p be an odd prime such that $p \equiv 1, 5, 19, 23 \pmod{24}$. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{28^{4k}} \equiv \begin{cases} 4x^2 \pmod{p} & \text{if } p \equiv 1, 19 \pmod{24} \text{ and so } p = x^2 + 18y^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 5, 23 \pmod{24}. \end{cases}$$

Proof. From [LM, Table II] we know that the elliptic curve defined by the equation $y^2 = x^3 + 4x^2 + (2 - \frac{40}{49}\sqrt{6})x$ has complex multiplication by the order of discriminant -72 . Thus, by (3.4) we have

$$\begin{aligned} & \sum_{n=0}^{p-1} \left(\frac{n^3 + 4n^2 + (2 - \frac{40}{49}\sqrt{6})n}{p} \right) \\ &= \begin{cases} 2x & \text{if } p \equiv 1, 19 \pmod{24} \text{ and so } p = x^2 + 18y^2, \\ 0 & \text{if } p \equiv 5, 23 \pmod{24}. \end{cases} \end{aligned}$$

Now taking $m = 28^4$ and $t = \frac{20}{49}\sqrt{6}$ in Lemmas 3.2 and 3.3 and applying the above we deduce the result.

Remark 3.4 Let $p > 7$ be a prime. Z.W. Sun conjectured that ([Su1, Conjecture A28])

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{28^{4k}} \equiv \begin{cases} 4x^2 - 2p \pmod{p} & \text{if } p = x^2 + 2y^2 \equiv 1, 3 \pmod{8}, \\ 0 \pmod{p^2} & \text{if } p \equiv 5, 7 \pmod{8}. \end{cases}$$

Theorem 3.10. *Let p be an odd prime such that $p \equiv \pm 1 \pmod{5}$. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{(-2^{14} \cdot 3^4 \cdot 5)^k} \equiv \begin{cases} 4x^2 \pmod{p} & \text{if } p = x^2 + 25y^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

Proof. From [LM, Table II] we know that the elliptic curve defined by the equation $y^2 = x^3 + 4x^2 + (2 - \frac{161}{180}\sqrt{5})x$ has complex multiplication by the order of discriminant -100 . Thus, by (3.4) we have

$$\sum_{n=0}^{p-1} \left(\frac{n^3 + 4n^2 + (2 - \frac{161}{180}\sqrt{5})n}{p} \right) = \begin{cases} 2x & \text{if } p = x^2 + 25y^2, \\ 0 & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

Now taking $m = -2^{14} \cdot 3^4 \cdot 5$ and $t = \frac{161}{360}\sqrt{5}$ in Lemmas 3.2 and 3.3 and applying the above we deduce the result.

Remark 3.5 Let $p > 7$ be a prime. Z.W. Sun made a conjecture ([Su1, Conjecture A25]) equivalent to

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{(-2^{14} \cdot 3^4 \cdot 5)^k} \equiv \begin{cases} 4x^2 - 2p \pmod{p} & \text{if } p = x^2 + 25y^2, \\ -4xy \pmod{p^2} & \text{if } p = x^2 + y^2 \text{ with } 5 \mid x - y, \\ 0 \pmod{p^2} & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

Theorem 3.11. *Let $p > 3$ be a prime. Then*

$$\begin{aligned} \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{648^k} &\equiv 0 \pmod{p^2} \quad \text{for } p \equiv 3 \pmod{4}, \\ \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{(-144)^k} &\equiv 0 \pmod{p^2} \quad \text{for } p \equiv 2 \pmod{3}, \\ \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{(-3969)^k} &\equiv 0 \pmod{p^2} \quad \text{for } p \equiv 3, 5, 6 \pmod{7}. \end{aligned}$$

Proof. This is immediate from Corollary 2.2 and [S2, Theorems 4.3-4.5].

We remark that Theorem 3.11 was conjectured by the author in [S1].

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